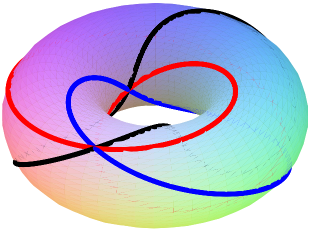

Affine descents and the Steinberg torus

T. Kyle Petersen
University of Michigan

(joint with Kevin Dilks and John Stembridge, arXiv:0709.4291)

Chicago, October 6, 2007

Affine descents and the Steinberg torus

Eulerian polynomials

The Eulerian polynomials, $A_n(t) = \sum_{k=0}^{n} a_{n,k} t^k$:

$$A_1(t) = 1 + t$$
$$A_2(t) = 1 + 4t + t^2$$
$$A_3(t) = 1 + 11t + 11t^2 + t^3$$
$$A_4(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$$
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- symmetric, unimodal coefficients
- real-rooted

Combinatorial interpretation

Let $d(w) := \#\{i : w_i > w_{i+1}\}$. Then,

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$$A_2(t) = 1 + 4t + t^2$$

A generalization

The notion of descent makes sense in any Coxeter system (W, S) (and simple roots Δ):

$$\begin{aligned} d(w) &:= \#\{s \in S : \ell(ws) < \ell(w)\} \\ &= \#\{\alpha \in \Delta : w(\alpha) < 0\} \end{aligned}$$

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- The W-Eulerian polynomials are symmetric, unimodal, (γ-nonnegative)
- Brenti has conjectured real-rootedness as well (D_n remains unproved)

Affine descents and the Steinberg torus

The f- and h-polynomials

Let Σ be a finite set of simplices, $f_k(\Sigma) =$ number of faces of dimension $k-1$

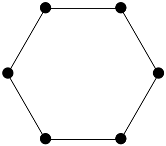
$$f(\Sigma;t) := \sum_{k=0}^{n} f_k(\Sigma)t^k$$

$(f_0, f_1, \ldots, f_n)$ is the *f-vector*

$$h(\Sigma;t) := (1-t)^n f(\Sigma; t/(1-t)) = \sum_{k=0}^{n} h_k(\Sigma)t^k$$

$(h_0, h_1, \ldots, h_n)$ is the *h-vector*

The f- and h-polynomials

Σ : 

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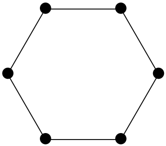
Σ : 

- $f_0 = 1$

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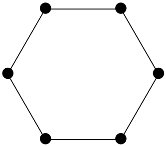
Σ : 

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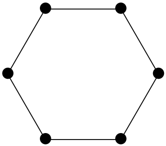
Σ : 

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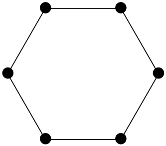
Σ : 

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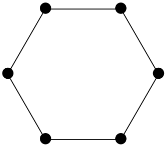
Σ : 

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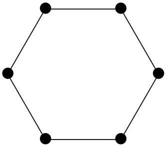
Σ : 

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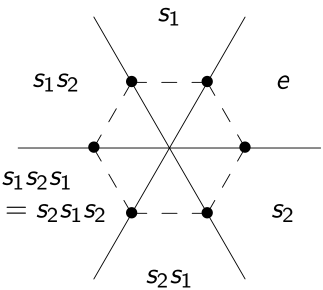
$$f(\Sigma; t) = 1 + 6t + 6t^2$$
$$h(\Sigma; t) = 1 + 4t + t^2 = A_2(t) \text{ (hmm...)}$$

The Coxeter complex

$\Sigma(A_2)$:



For a Coxeter system (W, S), the reflecting hyperplanes partition the ambient vector space

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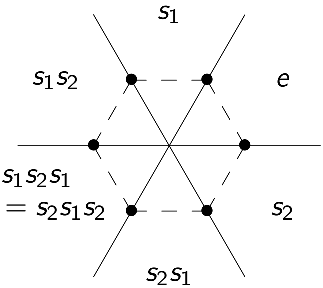

For a Coxeter system (W, S), the reflecting hyperplanes partition the ambient vector space

By intersecting the hyperplanes with the unit sphere we achieve a topological realization of the *Coxeter complex*, $\Sigma(W)$

The W-Eulerian polynomial

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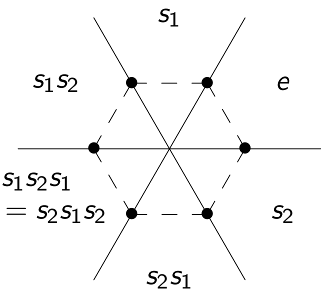

Theorem (Björner, Brenti)

For any finite Coxeter group W,

$$h(\Sigma(W); t) = \sum_{w \in W} t^{d(w)} = W(t)$$

Affine descents

If W is crystallographic, it has a unique lowest root $\alpha_0 = -\widetilde{\alpha}$

Let s_0 be the corresponding reflection, $\Delta_0 = \Delta \cup \{\alpha_0\}$, and

$$\begin{aligned} \widetilde{d}(w) &:= d(w) + \chi(\ell(ws_0) > \ell(w)) \\ &= \#\{\alpha \in \Delta_0 : w(\alpha) < 0\} \end{aligned}$$

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Definition (Dilks-Petersen-Stembridge)

The affine W-Eulerian polynomial is

$$\widetilde{W}(t) := \sum_{w \in W} t^{\widetilde{d}(w)}$$

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Results of D-P-S, $\widetilde{W}(t)$ is:

- γ-nonnegative ($\Rightarrow$ symmetric, unimodal)
- conjecturally real-rooted ($\widetilde{A}_n$, $\widetilde{C}_n$, exceptional groups are proved; $\widetilde{B}_n$ and $\widetilde{D}_n$ are verified for $n \leq 100$)

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$$\widetilde{A}_2(t) = 3t + 3t^2$$

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It is the topological construction whose h-vector is encoded by the affine Eulerian polynomial.

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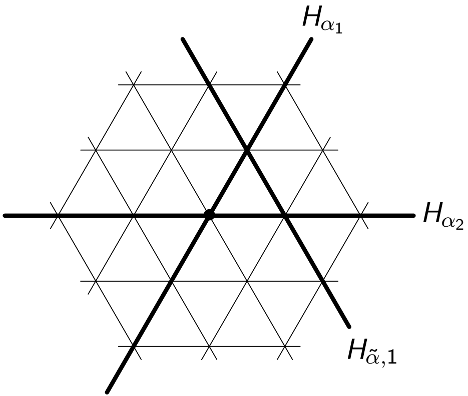
It is the topological construction whose h-vector is encoded by the affine Eulerian polynomial.

What is the Steinberg torus? (Correct!)

Affine Coxeter complexes

$\Sigma(\widetilde{A}_2)$:



The affine Weyl group $\widetilde{W}$ is generated by S along with the reflection through $H_{\widetilde{\alpha},1} := \{\lambda : \langle\widetilde{\alpha}, \lambda\rangle = 1\}$, drawing all hyperplanes gives $\Sigma(\widetilde{W})$ (…if W is irreducible…)

The Steinberg torus

Standard fact: the coroot lattice is a translation subgroup;

$$\widetilde{W} \cong W \ltimes \mathbb{Z}\Phi^\vee$$

Thus $\widetilde{W}$-action on V restricts to a W-action on the torus $V/\mathbb{Z}\Phi^\vee$ (Steinberg - looking at Poincaré series of affine Weyl group)

Definition (D-P-S)

The **Steinberg torus** *of* $\widetilde{W}$ *is*

$$\Sigma_T(\widetilde{W}) := \Sigma(\widetilde{W})/\mathbb{Z}\Phi^\vee$$

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- maximal cells of in bijection with elements of W

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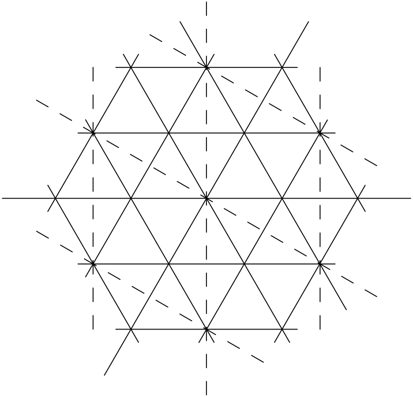
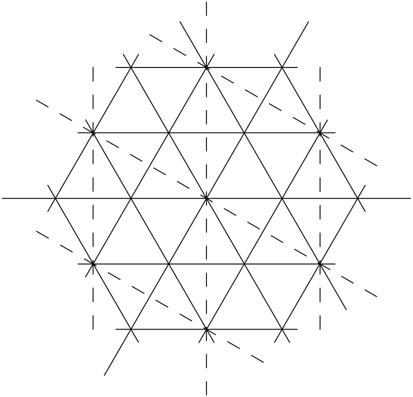


$$\Sigma_T(\widetilde{A}_2) := \Sigma(\widetilde{A}_2)/\mathbb{Z}\{\alpha_1^\vee, \alpha_2^\vee\}$$

The Steinberg torus



Equivalently, observe that exactly one vertex of every alcove is in $\mathbb{Z}\Phi^\vee$, so we translate can translate to the origin

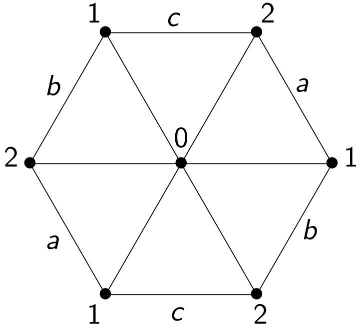
The Steinberg torus

The union of (closures of) the alcoves neighboring the origin is a convex, W-invariant simplicial polytope:

$$P_\Phi := \{\lambda \in V : -1 \leq \langle \lambda, \beta \rangle \leq 1 \text{ for all } \beta \in \Phi\}$$

We obtain the Steinberg torus by identifying opposite faces of P_Φ

$\Sigma_T(\widetilde{A}_2)$:



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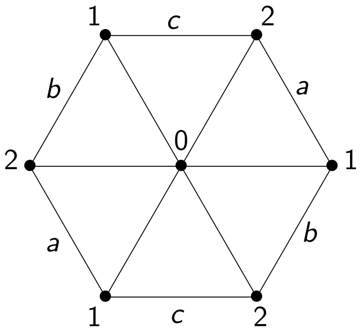

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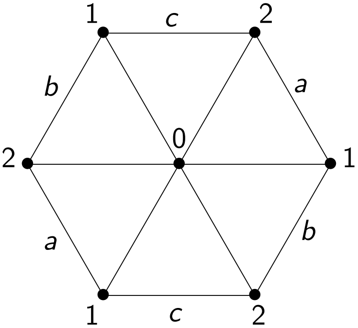

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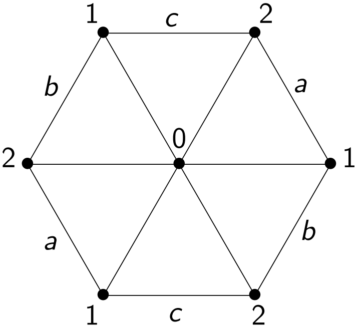

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The Steinberg torus

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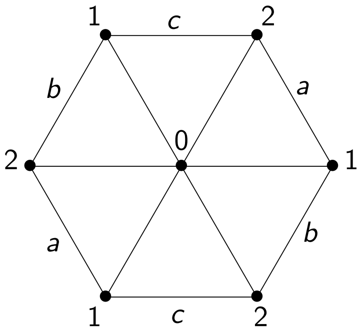

- $f_0 = 0$ (. . . if we ignore the empty face, things work out nicer. . .)
- $f_1 = 3$
- $f_2 = 9$

The Steinberg torus

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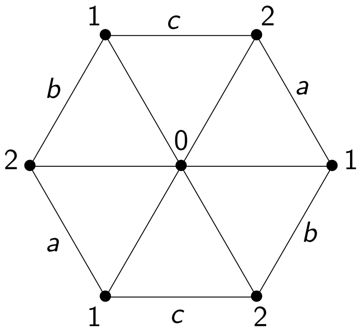

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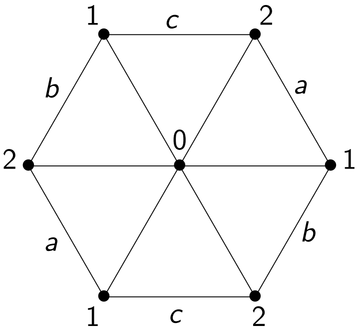

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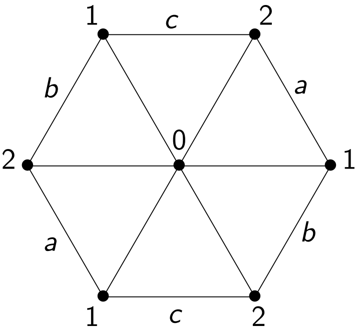

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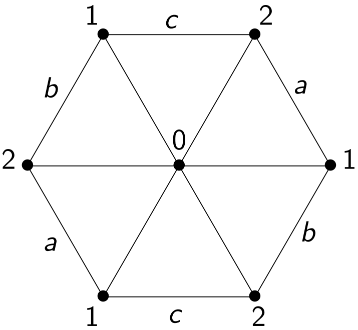

$f(\Sigma_T; t) = 3t + 9t^2 + 6t^3$ $\qquad$ $h(\Sigma_T; t) = 3t + 3t^2 = \widetilde{A}_2(t)$

The Steinberg torus

Theorem (D-P-S)

For any irreducible affine Weyl group $\widetilde{W} = W \ltimes \mathbb{Z}\Phi^\vee$,

$$h(\Sigma_T(\widetilde{W}); t) = \sum_{w \in W} t^{\widetilde{d}(w)} = \widetilde{W}(t)$$

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- General topological reasons to expect $h_i \geq 0$ here? (symmetry? unimodality? γ-nonnegativity?)

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- Reducible $\widetilde{W}$?

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- General topological reasons to expect $h_i \geq 0$ here? (symmetry? unimodality? γ-nonnegativity?)
- Real-rootedness: The remaining cases, $D_n, \widetilde{B}_n, \widetilde{D}_n$ are γ-nonnegative (a necessary condition for real roots in this situation)
  Each case boils down to peak combinatorics
- $\Sigma_T(\widetilde{C}_2)$ is barycentric subdivision of cube with opposite faces identified, has nonnegative cd-index: Is there a more general result?
- Reducible $\widetilde{W}$?
- "Fake" affine Eulerian polynomials

$$H_3^{fa}(t) = 26t + 68t^2 + 26t^3$$

$$H_4^{fa}(t) = 960t + 6240t^2 + 6240t^3 + 960t^4$$

Questions?

Art gallery:

http://www.math.lsa.umich.edu/~tkpeters/steinberg