I. VERY GENERALIZED RIEMANN DERIVATIVES

0. Generalized Riemann derivatives.

Let \( f \) be a real valued function of a real variable. The \( n \)th Riemann derivative of \( f \) is

\[
R_n f(x) := \lim_{h \to 0} \frac{\sum_{i=0}^{n} (-1)^{n-i} f(x + (-\frac{n}{2} + i)h)}{h^n}
\]

The first two special cases

\[
R_1 f(x) = \lim_{h \to 0} \frac{-f(x - \frac{h}{2}) + f(x + \frac{h}{2})}{h}
\]

and

\[
R_2 f(x) = \lim_{h \to 0} \frac{f(x-h) - 2f(x) + f(x + h)}{h^2}
\]

are the well known symmetric and Schwarz derivatives.

The generalized Riemann derivative which was the subject of my 1966 thesis[1] is

\[
D_n(b,a) f(x) := \lim_{h \to 0} \frac{A_n(h;b,a) f(x)}{h^n}
\]

where

\[
A_n(h;b,a) f(x) := \sum_{i=0}^{n+e} a_i f(x + b_i h)
\]

[1] The research presented here was supported in part by a grant from the Faculty Research and Development Fund of the College of Liberal Arts and Sciences, DePaul University
where e is a non-negative integer which I will call the excess and the \( a_i \)'s and \( b_i \)'s are real numbers. Here we insist upon the \( n+1 \) consistency conditions

\[
I a_i b^j = \begin{cases} 0 & j = 0, 1, \ldots, n-1 \\ n! & j = n \end{cases}.
\]

For notational convenience I will always assume \( b_0 < b_1 < \ldots < b_{n+e} \).

1. **Relations between different generalized derivatives.**

To see why these conditions are imposed let \( f^{(n)}(x_o) \) exist so that

\[
f(x_o + k) = \sum_{j=0}^{n} \frac{f^{(j)}(x_o)}{j!} k^j + o(k^{n+1}).
\]

(Here and throughout \( g(h) = o(h^\alpha) \) means \( \frac{g(h)}{h^\alpha} \to 0 \) as \( h \to 0 \).) This expansion is a slightly souped up version of Taylor's theorem which is due to de la Vallee-Poussin. Professor A. Zygmund showed it to me. Substitute this into (1) with \( k \) equal successively \( b_0 h, b_1 h, \ldots, b_{n+e} h \) to get

\[
\sum a_i f(x_o + b_i h) = \sum a_i \left[ \sum \frac{f^{(j)}(x_o)(b_i h)^j}{j!} \right] + o(h^n)
\]

\[
= \sum \frac{f^{(j)}(x_o)}{j!} h^j \left[ \sum a_i b_i^j \right] + o(h^n)
\]

\[
= \frac{f^{(n)}(x_o)}{n!} [n!] h^n + o(h^n).
\]

Divide by \( h^n \) and let \( h \to 0 \). We get \( D_n f(x_o) \) so that our derivatives are extensions of the usual ones. Very simple examples show these extensions to be strict. For example, \( a(x) = |x| \) has \( R_1 a(0) = 0 \) while \( a'(0) \) does not exist, and \( s(x) = \text{signum}(x) \) has \( R_2 s(O) = 0 \) while \( s'(0) \) and \( s''(0) \) do not exist.
The reason for calling $e$ the excess is that if $e=0$ then the $b_i$'s determine the $a_i$'s via condition (2). Explicitly,

\[ a_i = \frac{n!}{\prod_{j \neq i} (b_i - b_j)} \frac{\prod_{j \neq i} (x - b_j)}{\prod_{j \neq i} (b_i - b_j)} \]

To see this, let $L_i(x) := \frac{\prod_{j \neq i} (x - b_j)}{\prod_{j \neq i} (b_i - b_j)}$ be the Lagrange interpolating polynomial so that $L_i(b_i) = 1$ and $L_i(b_j) = 0$ when $j \neq i$. Then from (2) it is immediate that $A_n(1;b,a)L_i(0) = a_i$. On the other hand, $L_i(x) = [\prod(b_i - b_j)]^{-1}x^n + \text{lower powers of } x$, whence the nth ordinary derivative of $L_i$ is the constant $n!/[\prod(b_i - b_j)]^{-1}$.

The Taylor expansion out to $h^n$ is exact, i.e., without higher order terms, for polynomials of degree $n$, so that equations (4) show that

\[ A_n(h;b,a)L_i(x) \]

is equal to this constant. Setting $x = 0$ and $h = 1$ proves (5). In particular, you can't make a first derivative without at least 2 terms, nor a second without at least 3, nor an $n$-th without at least $n+1$ points.

On the other hand even if all $b_i$'s are fixed, if $e > 0$ you can choose $e$ of the $a_i$'s freely; then conditions (2) determine the rest.

Denjoy looked at the case of excess $= 0$.[11] I seem to have been the first to look at $e > 0$ systematically although particular cases have shown up in numerical analysis before.

The $n$-th Peano derivative $f_n$ is a generalization of the ordinary derivative lying midway between the ordinary $n$-th
derivative and \( D_n f(x) \). By definition \( f_n(x) \) exists if \( n \) other numbers \( f_0(x), f_1(x), \ldots, f_{n-1}(x) \) also exist so that
\[
f(x_c + h) = f_0(x) + f_1(x)h + \ldots + f_{n-1}(x) \frac{h^n}{n!} + o(h^n).
\]
Note that \( f \) is continuous at \( x \) if \( f_0(x) = f(x) \) and \( f \) is differentiable at \( x \) if and only if \( f_1(x) \) exists. Then \( f'(x) = f_1(x) \).

The classic example showing \( f_2 \) to be a strict extension of \( f'' \) is \( x^3 \sin \frac{1}{x} \) at the point \( x=0 \). Note that what we proved above shows each \( D_n \) to be an extension of \( f_n \). Also note that the examples \( a(x) \) and \( s(x) \) show \( R_1 \) a strict extension of \( f_1(=f') \) and \( R_2 \) a strict extension of \( f_2 \). Again every \( D_n \) (except \( D_1 \) with \( a_0=0, a_1=1 \)) is a strict extension of the corresponding \( f_n \).

However the implication \( \exists f_n \rightarrow \exists D_n \) is reversible provided we are willing to throw away a set of Lebesgue measure 0. This was the main result of my 1966 PhD thesis.\([1]\)

If \( n \geq 2 \), one cannot return from \( f_n \) to \( f^{(n)} \) even on an almost everywhere basis. This question was discussed by Oliver in 1953. \([15]\) He does prove that \( \exists f_n \rightarrow \exists f^{(n)} \) provided \( f_n(x) \) is a bounded function on an interval as well as several other interesting results.

There is also a derivative, designated \( d_2 \) in \([2]\), which lies between \( f_2 \) and every \( D_2 \) in an almost everywhere sense.

Most of these notions and results go through in an \( L^P \) metric sense.\([1],[2]\)
Another way to return from $D_n$ to $f_n$ does work at a single point. This time assume that $f$ is measurable and that every $D_n f(x_0)$ exists. Then it does follow that $f_n(x_0)$ exists. To improve on this result one should cut down on the number of Riemann derivatives assumed existent at $x_0$. Coupling the results of a 1969 paper — A Characterization of the Peano derivative — and a 1974 paper with Erdos and Rubel we have the following result. [2],[5]

Let $d_1(h) := f(x+h)-f(x)$,

$$d_2(a_1,h) := d_1(a_1h) - a_1d_1(h) = f(x+a_1h) - a_1f(x+h) + (a_1-1)f(x), \ldots$$

$$d_n(a_1,\ldots,a_{n-1};h) := d_{n-1}(a_1,\ldots,a_{n-2};a_{n-1}h) - a_{n-1}d_{n-1}(a_1,\ldots,a_{n-2};h)$$

and let $D_n(a)(x) := \lim_{h \to 0} \frac{d_n(a;h)}{h^n}$ (The $a_i$'s are not 0, 1 or -1.) If $f$ is measurable, and if whenever $a \in M^{n-1}$, $D_n(a)$ exists at $x = x_0$, and if $M$ is "thick" enough; then $f_n(x_0)$ exists. The thickness of the set $M$ determines how good this theorem is. Easy examples show that it is not enough for $M$ to be countably infinite, nor for $M$ to consist solely of positive numbers. If $M$ has positive measure and contains a negative number then $M$ is thick enough.

At $x=0$ the second derivative $R_2$ differentiates $s(x)$ but not $a(x)$, while the second derivative

$$P_2f(x) := \lim_{h \to 0} \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2}$$

does not differentiate $s(x)$, but does differentiate $a(x)$ since looking only forward $a(x)$ is a straight line and looking only backwards $a(x)$ is also a straight line. However Patrick J. O'Connor, in an unpublished 1969 PhD
thesis at Connecticut Wesleyan shows that whenever two generalized Riemann n-th derivatives both exist at a point, they must agree.[14]

The idea of his proof is quite nice. If \( D_n = \lim_{h \to 0} \sum_{i,j} a_i f(x+b_i h) \)
and \( D'_n = \lim_{h \to 0} \sum_{i,j} a'_i f(x+b'_i h) \), form \( D_n \odot D'_n := \lim_{h \to 0} \sum_{i,j} a_i a'_j f(x+b_i b'_j h) \).
It is then easy to prove that \( D_n \odot D'_n \) is also a generalized Riemann derivative and that it agrees with both \( D_n \) and \( D'_n \).

2. **Numerical Analysis.**

Generalized Riemann derivatives have had application in numerical analysis. The symmetric derivative \( R_1 \) is "better" for approximation purposes than the ordinary derivative in the sense that for fixed \( h \) and very smooth \( f \),
\[
\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{1}{2} f''(t) h \quad \text{while} \quad \frac{f(x+\frac{h}{2}) - f(x-\frac{h}{2})}{h} = f'(x) + \frac{1}{48} f^{(3)}(t) h^2
\]
and the error term \( \frac{1}{48} f^{(3)}(t) h^2 \) is "sort of smaller" than \( \frac{1}{2} f''(t) h \). Notice that to make the comparison fair I normalize and keep \( b_2 - b_1 = 1 \) in both cases. So to compare approximations to the first derivative based on 2+e function evaluations I fix \( h \) and look at differences
\[
\sum_{i=0}^{e+1} a_i f(x+b_i h) = A(b,a) f(x) \quad \text{subject to this normalization}
\]
\( b_{i+1} - b_i \geq 1 \) for all \( i \geq 0 \). If 2 such differences give for good \( f \)
\[
A(b,a) f(x) = f'(x) + c_r f^{(r)}(x) h^{r-1} + o(h^r)
\]
and
\[
A(b',a') f(x) = f'(x) + c_s f^{(s)}(x) h^{s-1} + o(h^s)
\]
define \( A(b,a) \) to be better than \( A(b',a') \) if either \( r > s \), or \( r=s \) and \( c_r < c_s \).
Then indeed \( b = (-\frac{1}{2}, \frac{1}{2}) \) gives the best 2 point difference. Again the best 4 point difference has \( b = (-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}) \) which is still no surprise. Again the answer you would guess for 6, 8, or any even number of points is correct. However, for 3 points the best \( b \) is
\[
b = \left( \frac{1}{\sqrt{3}}, -1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, +1 \right) \approx (-.423, .577, 1.577),
\]
\[
= (a_3-1, a_3, a_3+1),
\]
for 5 points
\[
b = (a_5-2, a_5-1, a_5, a_5+1, a_5+2)
\]
where \( a_5 = \sqrt{15 - \sqrt{145}}/10 \approx .544 \), and for \( 2k+1 \) points
\[
b = (a_{2k+1} - k, \ldots, a_{2k+1}, \ldots, a_{2k+1} + k) \]
where the \( a_n \) satisfy \( \frac{1}{2} < a_n < \frac{1}{2} + \frac{1}{4n}, n=3,5,\ldots \) and \( a_n \) is determined as the smallest positive zero of \( \frac{d}{dx} \left( \pi (x-1) \right) = 0 \). The choice of \( b \) and the approximating conditions
\[
\sum a_i = 0 \\
\sum a_i b_i = 1 \\
\sum a_i b_i^j = 0 \quad j = 2,3,\ldots,n-2
\]
determine \( a \) by linear algebra. This choice is unique up to the trivial inversion \( (b,a) \rightarrow (-b,-a) \).

A similar situation occurs for the second derivative. Here the starting point is that \( R_2 \) gives the best 3 point difference. The results are similar to those above. Now the best 3,5,7,\ldots point
differences are based on the obvious symmetric choices of \( b \) while the even \( b \)'s are more interesting with the best 4 point \( b \) being

\[
b = (\beta_4, \beta_4^{-1}, \beta_4, \beta_4^{-1}), \quad \beta_4 = (1 + \sqrt{5/3})/2 \approx 1.145 \text{ and so on.}
\]

In a 1981 Math. Comp. paper Roger Jones and I work out the 3 point first derivative case which remains optimal even when roundoff error is taken into account [7]. The general results I just mentioned are detailed in a 1984 paper in Estratto de Calcolo with Svante Janson and Roger Jones.[9]

**Question 1.** Extend these results to \( n > 2 \). (Even \( n=3 \) was too hard for us.)

3. **Classification Questions**

A very interesting example is provided by the first derivative

\[
0_1 f(x) := \lim_{h \to 0} \frac{7f(x+3h) - 13f(x+4h) + 6f(x+\frac{16}{3}h)}{h}
\]

and the function

\[
f(x) := \text{sgn}(x)|x|^{\log_{4/3}(7/6)} - x.
\]

This example is given by Patrick O'Connor in his thesis.[14] Since

\[
p := \log_{4/3}(7/6) = \frac{\ln(7/6)}{\ln(4/3)} \approx .54, \ \text{sgn}(x)|x|^p \text{ looks like } \text{sgn}(x)^{\frac{1}{|x|}}.
\]
and $f$ looks about the same. But then $0_1 f(x) = f'(x)$ whenever $x \neq 0$ and direct calculation shows that $0_1 f(0) = -1$. This example has a lot of shock value for me. Here is the graph of $0_1$.

We have a non-Darboux derivative. We also have an everywhere increasing, everywhere differentiable (with respect to $0_1$) function whose derivative is negative at a point.

On the other hand consider the symmetric derivative $R_1$. This derivative's existence does force a function to be Darboux. If a strictly increasing function has an everywhere existing symmetric derivative, then that derivative is positive. These two properties also hold for $f'$. We thus have at least 2 classification problems.

**Question 2.** Which generalized Riemann derivatives are Darboux? That is, for which $D_1$ does the existence of $D_1 f(x) = f(x)$ at every point $x$ force $f$ to have the intermediate value property?

**Question 3.** For which $D_1$ does $f$ increasing on $(a-\epsilon, a+\epsilon)$ and $D_1 f(a)$ existing force $D_1 f(a) > 0$?

Notice that for both questions $0_1$ is in the bad class, while $R_1$ and $\frac{d}{dx}$ are both in the good class.

4. **Further generalization.**

Let us now justify the "very" in the title of the talk. By the very generalized Riemann derivative $D^+_n(b,a)$ I mean the same thing as before except that the limit is now one sided, so

$$D_n(b,a) f(x) = \lim_{b+0^+} \frac{D_n(h;b,a)f(x)}{h^n}.$$
There is no need for a $D_n^-$ to be defined since for example one has
\[
\frac{\sum_i f(x + b_i h)}{h} = \lim_{h \to 0^-} \frac{\sum(-a_i)f(x + (-b_i)(-h))}{h} = \lim_{h \to 0^+} \frac{\sum(-a_i)f(x + (-b_i)h)}{h} = D_1^+(-b, -a).
\]

One could go on to define objects similar to Dini numbers such as
\[
\lim \sup_{h \to 0^+} \frac{A_n(h; b, a)f(x)}{h^n}
\]
but I have not done anything in this direction.

It is obvious that $D_n^+$ is an extension of $D_n$, i.e. that if $D_n(b, a)f(x_0)$ exists so does $D_n^+(b, a)f(x_0)$ and the two are then equal. The extension is usually proper. Note that $R_n^+ = R_n$ and more generally enough symmetry in $a$ and $b$ will make $D_n^+ = D_n$. Probably one could prove that $\{(b_i, a_i)\} = \{(-b_i, -a_i)\}$ for $n$ odd and $\{(b_i, a_i)\} = \{(-b_i, a_i)\}$ for $n$ even is a necessary and sufficient condition for the extension to be improper, i.e., for $D_n^+ = D_n$ to hold.

The function $a(x) = |x|$ has $\left(\frac{d}{dx}\right)^+ a(0) = 1$ although $\left(\frac{d}{dx}\right)^- a(0)$ doesn't exist. A more interesting example is the second derivative
\[
A_2^+(x) := \lim_{h \to 0^+} \frac{(2/3)f(x+2h) - f(x+h) + (1/3)f(x-h)}{h^2}. \text{ Note that } \frac{2}{3} - 1 + \frac{1}{3} = 0, \frac{2}{3}(-1) + \frac{1}{3}(-1) = 0 \text{ and } \frac{2}{3}(2)^2 - 1(1)^2 + \frac{1}{3}(-1)^2 = 2.
\]
Then consider the function $u(x) = \begin{cases} 0 & x < 0 \\ \log_2(3/2) & x \geq 0 \end{cases}$. For $h > 0$,
\[
\frac{(2/3)u(0+2h) - u(0+h) + (1/3)u(0-h)}{h^2} = \frac{\log_2(3/2)}{h^2} \cdot \left(1 - \frac{1}{h^{2-q}}\right) = 0,
\]
so that $A_2^+ u(0) = 0$.  

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Clearly for $x \neq 0$, $A_2^+ u(x) = u''(x) = \begin{cases} 0 & x < 0 \\ q(1-q)x^{q-2} & x > 0 \end{cases}$. A similar calculation for $h < 0$ shows that $A_2 u(0)$ does not exist.

Again $q := \log_2(\frac{3}{2}) = \frac{\ln(3/2)}{\ln 2} \approx .58$ so $\log_2(3/2)$ looks like $\sqrt{x}$ for positive $x$. Here is $u$.

If one allows $h \to 0^-$ as well, then the situation of continuous non-convex $f$ with $A_2 f \geq 0$ everywhere does not arise. One reason to study $A_2^+$ is the following. The 0 excess very generalized second Riemann derivatives may be classified as

- type I if $b_0 < b_1 = 0 < b_2$ ;
- type II if $b_0 < 0 < b_1 < b_2$ or if $b_0 < b_1 < 0 < b_2$ ; and
- type III if $b_0 < b_1 < b_2 < 0$ or if $0 \leq b_0 < b_1 < b_2$ .

I think that all the questions I will raise in studying $A_2^+$ will have easy answers for type I and type III derivative and that $A_2^+$ will prove to be a prototype for all those of type II. We will see more of $u$ and $A_2^+$ shortly.
II. GENERALIZED RIEMANN DERIVATIVES AND ASSOCIATED SUMMABILITY METHODS

5. Generalized differentiation and uniqueness for trigonometric series.

Let \( T = \sum c_n e^{inx} \) be a trigonometric series. Suppose that at every \( x \in [0, 2\pi) \), \( T(x) := \lim_{N \to \infty} \sum_{-N}^{N} c_n e^{inx} = 0 \). Then all \( c_n = 0 \). This is the fundamental theorem in the subject. It was announced by Riemann in 1854 and the last detail of his proof was supplied in a letter from H.A. Schwarz to Cantor who published it in 1870. [10],[16],[17]

Theorem R. If \( F \) is continuous and \( R_2 F = 0 \) everywhere, then \( F \) is a line.

This theorem is immediate from a lemma.

Lemma R. If \( F \) is continuous and \( R_2 F \geq 0 \) everywhere then \( F \) is convex.

Consider the following statement.

"Lemma" A. If \( F \) is continuous and \( A^+_2 F \geq 0 \) everywhere, then \( F \) is convex.

As the continuous non-convex \( u \) enjoys \( A^+_2 u \geq 0 \) for all \( x \), this statement is false.

However, we are left with the following open question.

"Theorem" A. If \( F \) is continuous and \( A^+_2 F = 0 \) everywhere, then \( F \) is linear.

Question 4. Is "Theorem" A true?
This question is very hard. Why does it matter? On the one hand, theorem R is the cornerstone of the entire theory of uniqueness. There are many open questions concerning multiple trigonometric series whose resolution would be easy if higher dimensional analogues of Theorem R were available. For example suppose \( T(x,y,z) \) converges unrestrictedly rectangularly to 0, that is, suppose

\[
\lim_{L,M,N \to \infty} \lim_{l=-L}^{L} \lim_{m=-M}^{M} \lim_{n=-N}^{N} c_{l,m,n} e^{i(lx + my + nz)} = 0, \text{ at every } (x,y,z).
\]

No one knows if it then follows that all \( c_{l,m,n} \) are 0. On the other hand, Theorem R has only one known proof, namely via Lemma R. To extend Theorem R to higher dimensional settings it could be useful to have another proof. A proof of "Theorem" A couldn't use the false "Lemma" A and so would probably also yield a genuinely new proof of Theorem R.

Another question related to uniqueness is

**Question 5.** Let \( F(x,y) \) be continuous and suppose

\[
0 = \lim_{h,k \to 0} \left\{ \frac{F(x-h,y+k) - 2F(x,y+k) + F(x+h,y+k)}{h^2} \right\} \cdot \frac{1}{h^2}
\]

at each \((x,y)\). Is \( F \) then necessarily of the form \( F(x,y) = (ax + b) + (cy + d) \) where \( a \) and \( b \) are functions of only \( y \), and \( c \) and \( d \) are functions of only \( x \)? See my paper with Welland or my survey article in my book for some details and partial results about this.\[3],[6\]

A related question is

**Question 6.** It follows easily from Theorem R that if

\[
\frac{1}{h} \int_{0}^{h} |f(x+t) - f(x-t)| \, dt = o(h) \quad \text{at all points } x, \text{ then } f \text{ is constant.}
\]

Prove this without invoking Lemma R.
This would follow if a function with everywhere 0 symmetric approximate derivative could be shown to be constant. A positive resolution of question 6 will necessarily also provide a new proof of Riemann's uniqueness theorem.[4]

6. **Generalized Differentiation and Summability.**

In an attempt to prove "Theorem" A I was led to a related summability result. Let \( F(x) = \sum c_n e^{inx} \) be a continuous function. Form the distributional second derivatives \( F'' := \sum (in)^2 c_n e^{inx} \). An elementary computation shows

\[
\frac{F(x+h) - 2F(x) + F(x-h)}{h^2} = \sum (in)^2 c_n e^{inx} \left( \frac{\sin nh}{nh} \right)^2.
\]

By definition \( R_2 F(x) := \lim_{h \to 0} \) (L.H.S.) and by definition the series \( F'' \)

is summable \((R, 2)\) to \( s \) if \( s = \lim_{h \to 0} \) (R.H.S.). Thus theorem \( R \) can be restated by saying that a continuous function whose distributional second derivative is summable \((R, 2)\) everywhere to 0 is linear.

Similarly the derivative \( A_2^+ \) corresponds to a method of summability, call it summability \( A_2^+ \). There is a theorem of Kuttner [13] that summability \((R, 2)\) implies Abel summability and a theorem of Verblunsky [17] stating that if \( \sum c_n e^{inx} \) is Abel summable to 0 everywhere and \( c_n = o(n) \) then all \( c_n = 0 \). I hoped to show "Theorem" A by first showing summability \( A_2^+ \) implies Abel summability, then controlling the coefficients, and finally applying Verblunsky's theorem.
So define a series \( \sum a_n \) to be summable \( A^+_2 \) to \( s \) if
\[
\lim_{n \to \infty} \sum_{n=1}^{\infty} a_n F(\text{inh}) = s \quad \text{where}
\]
\[
F(t) = \frac{(2/3)e^{2t} - e^t + (1/3)e^{-t}}{t^2}.
\]
As with the Riemann situation we have \( A^+_2 F(x) \) exists if and only if the twice formally differentiated Fourier series of \( F \) is summable \( A^+_2 \). The function \( u(x) \) above, restricted to \([\pi, \pi]\) and then extended periodically, thus has \( u'' \), its distributional second derivative, summable \( A^+_2 \) to 0 at 0. However \( u'' \) is not Abel summable at 0 as a direct calculation shows so summability \( A^+_2 \) does not imply Abel summability.

7. **Mean Value Theorems for Generalized Riemann Derivatives.**

The prettiest type of mean value theorem would say something like this. Let \( I = [x+b, x+b+e] \) where \( x \) and \( b \) are fixed. If \( \Delta_n f(t) \) exists for every \( t \in I \), then there is a \( \xi \) interior to \( I \) with
\[
\frac{\Delta_n(h; b, a)f(x)}{h^n} = \Delta_n f(\xi).
\]

But this is not even true for \( R_1 \) as the choices \( x = -1 \), \( h = 3 \) and \( f(t) = |t| \) show.
I would suspect that the only generalized Riemann derivative for which this mean value theorem holds is \( \frac{d}{dx} \) itself.

**Question 7.** Classify the \( D_n \) for which the mean value theorem in the above form is true.

A more fruitful set of mean value theorems are those of following type.

Statement M(\( b, a \)). Fix \( x \) and \( h \) and set \( I = [x + b_0 h, x + b_{n+1} h] \). If \( f^{(n-1)}(t) \) is continuous on \( I \) and differentiable for all \( t \) interior to \( I \), then there is a \( t \) interior to \( I \) with \( \frac{\Delta_n(h; b, a)f(x)}{h^n} = f^{(n)}(t) \).

A classification of the set of \( (b, a) \) for which this statement is true is the goal of my present research with Roger Jones who is also at DePaul.[8]

We have a sufficient condition which is totally operational and which we can show to be necessary for all first and second generalized Riemann derivatives.

Let \( p_0, \ldots, p_e \) be real numbers with \( \sum p_i = 1 \). Let \( b_0 < b_1 < \ldots < b_{n+1} \) be \( n+1+e \) real numbers. Let \( D_0 \) be the unique generalized \( n \)-th derivative based on \( \{b_0, \ldots, b_n\} \), \( D_1 \) the unique one based on \( \{b_1, \ldots, b_{n+1}\}, \ldots, D_e \) the unique one based on \( \{b_e, \ldots, b_{n+e}\} \), and set \( D = \sum_{i=0}^{e} p_i D_i \). Then a quick check of the consistency condition shows that \( D \) is also an \( n \)-th derivative. Conversely, given any \( n \)-th generalized Riemann derivative \( D \) based on \( \{b_0, \ldots, b_{n+e}\} \) we can write \( D \) as \( \sum p_i D_i \) where the \( p_i \) are uniquely determined by \( b \) and \( a \). The \( p_i \) are very easily found and satisfy \( \sum p_i = 1 \).
For example, O'Connor's derivative is associated to

\[
\frac{7f(x+3h)-13f(x+4h)+6f(x+(16/3)h)}{h} = \\
7 \left[ \frac{f(x+3h)-f(x+4h)}{h} \right] - 6 \left[ \frac{f(x+4h)-f(x+(16/3)h)}{h} \right] = \\
-7 \left[ \frac{-f(x+3h)+f(x+4h)}{h} \right] + 8 \left[ \frac{-f(x+4h)+f(x+(16/3)h)}{(4/3)h} \right].
\]

So letting \(D_0\) and \(D_1\) be the limits of the last 2 bracketed expressions, as \(h \to 0\) we have \(O_\omega = p_0D_0 + p_1D_1\),

where \(p_0 + p_1 = -7 + 8 = 1\).

**Theorem.** Let \(D_n(b,a)\) be an \(n\)-th generalized Riemann derivative.

i) If the \(p_i\) associated to \(D\) are all positive (so that \(D\) is a convex combination of \(n\)-th derivatives without excess), then Theorem \(M(b,a)\) holds.

ii) Conversely if \(n=1\) or \(n=2\) or \(e=1\), and if any \(p_i\) is negative; then Statement \(M(b,a)\) is false.

**Question 8.** What happens if \(n>3\), \(e>2\), and some \(p_i\) is negative?

In particular, what happens for the excess 2 third derivative

\[D := (5/8)D_0 - (1/4)D_1 + (5/8)D_2,\] where for \(i = 0, 1, 2,\)

\[D_i := -f(x+ih)+3f(x+[i+1]h)-3f(x+[i+2]h)+f(x+[i+3]h)?\]

The proof of i) is short and sweet. First if \(e=0\) then \(p_0=1\) and indeed Theorem \(M\) is a well established numerical analysis fact.[12]

If \(e>0\), using this fact \(e+1\) times we have numbers \(\xi_i\) so that

\[s = \frac{D_n(h; b, a)f(x)}{h^n} = \sum_{i=0}^{e} p_i f^{(m)}(\xi_i).\]
The right side is a convex combination of the numbers 
\( f^{(n)}(x_0), \ldots, f^{(n)}(x_e) \) and hence \( s \) lies between the smallest and the largest. But \( f^{(n)} = (f^{(n-1)})' \) is an ordinary first derivative, hence is Darboux and therefore assumes the value \( s \).

The proof of ii) is longer so we will restrict ourselves to one simple case. Let \( b_0 < b_1 < b_2 \), let \( A_0 \) be the difference quotient associated to the unique first derivative based on \( \{b_0, b_1\} \), \( A_1 \) the one based on \( \{b_1, b_2\} \), and \( A = -7A_0 + 8A_1 \). Let \( f \) be this piecewise linear function.

Then \( A_1 = 1, A_0 = 0 \) so \( A = 8 \), but \( f' = 0 \) or \( 1 \). Finally round the corner at \( b_1 \) very slightly. This will make \( \text{Range}(f') = [0, 1] \) but keep \( A \) close to 8 so that the mean values theorem fails for \( A \).

We do the second derivative case by piecing together quadratics and then rounding the corners. The example for the general \( n \), excess 2 derivative case uses an \( n \)th degree polynomial.
REFERENCES


